On the complexity of ECDLP for composite fields

Based on joint works

with JC Faugère, JJ Quisquater, L Perret, G Renault

Christophe Petit

Discrete logarithm problem (DLP)

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  Given G a finite (multiplicative) cyclic group
  Given g a generator of G and given $h \in G$
  Find $k \in \mathbb{Z}$ such that $g^k = h$
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- Diffie-Hellman key exchange, ElGamal encryption, Digital Signature algorithm,...
- Cryptographic **assumption** : DLP is "hard" for
  - Multiplicative groups of finite fields
  - Elliptic curves
  - Jacobians of hyperelliptic curves

How hard is DLP ?

- Answer **depends on the group**
  - Subexponential algorithms exist for finite fields and hyperelliptic curves
  - Particular elliptic curve families are weaker
  - 160-bit ECDLP $\approx$ 2048-bit DLP or factoring

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  - Includes 10/15 curves standardized by NIST
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  - Complexity thought to be exponential in n
  - We argue it is

$$\leq 2^{2n^{2/3}\log n}$$

Outline

From ECDLP to polynomial systems

Algebraic cryptanalysis

Polynomial systems arising from a Weil descent

Back to ECDLP

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ECDLP on binary curves

- **Elliptic curve discrete logarithm problem**
  Given E over a finite field K,
  Given $P \in E(K)$, given $Q \in G :=< P >$,
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- **Binary curves** $K = \mathbb{F}_{2^n}$

$$y^2 + xy = x^3 + a_2x^2 + a_6 \quad \text{with } a_6 \neq 0$$

  Koblitz curve if $a_6 = 1$ and $a_2 \in \{0, 1\}$

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- **How hard is ECDLP on binary curves ?**

Generic DLP attacks

- Some attacks apply to DLP for any group G
  - Exhaustive search
  - Baby-step, giant step
  - Pollard's rho
  - Pohlig-Hellman if $|G|$ is smooth

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- In general, no better algorithm for elliptic curves
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  Transfer ECDLP to the Jacobian of an hyperelliptic curve
- Only work for specific families

This talk : Index calculus

- General method to solve discrete logarithm problems
  1. Define a **factor basis** $\mathcal{F} \subset G$
  2. **Relation search** : find about $|\mathcal{F}|$ **relations**

$$a_i P + b_i Q = \sum_{P_j \in \mathcal{F}} e_{ij} P_j$$

  3. Do **linear algebra** modulo $|G|$ on the relations to get

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- Define $\mathcal{F}$ s.t. there is an "efficient" algorithm for Step 2
- Balance relation search and linear algebra

Example : a naive index calculus for $\mathbb{F}_{2^n}^*$

- DLP : given $g, h \in \mathbb{F}_{2^n}^*$, find k such that $h = g^k$
- Factor basis made of **small "primes"**

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- For $B \approx n^{1/2}$, we get **subexponential complexity**

Index calculus : success stories

- **Finite fields :** Adleman [A79,A94], Coppersmith [C84], Adleman and Huang [AH99]
  Subexponential complexity

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  Adleman-DeMarrais-Huang [ADH94], Gaudry [G00], Gaudry-Thomé-Thériault-Diem [GTTD07]
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- **Elliptic curves :** no algorithm at all until 2005

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- First partial solutions given by Semaev [S04]

Summation polynomials [S04]

- Relate the x-coordinates of points that sum to O
- $S_r(x_1, \ldots, x_r) = 0$
  $\Leftrightarrow \quad \exists (x_i, y_i) \in E \quad \text{s.t.} \quad (x_1, y_1) + \cdots + (x_r, y_r) = O$

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- Recursive formulae :
  $S_2(x_1, x_2) = x_1 - x_2$
  $S_3(x_1, x_2, x_3) = \ldots$ (depends on E)
  $S_r(x_1, \ldots, x_r) =$
  $Res_X\left(S_{r-k}(x_1, \ldots, x_{m-k-1}, X), S_{k+2}(x_{r-k}, \ldots, x_r, X)\right)$

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  $Res_X\left(S_{r-k}(x_1,\ldots,x_{m-k-1},X),S_{k+2}(x_{r-k},\ldots,x_r,X)\right)$
- S_r has degree 2^{r-2} in each variable
  Symmetric set of solutions

Semaev's variant of index calculus

- Semaev's variant of index calculus :
  - **Factor basis** :
    define $\mathcal{F}_V := \{(x, y) \in E | \mathbf{x} \in \mathbf{V}\}$ where $V \subset K$
  - **Relation search** : for each relation,
    Compute $(X_i, Y_i) := a_i P + b_i Q$ for random a_i, b_i
    Find $\mathbf{x_j} \in \mathbf{V}$ **with** $\mathbf{S_{m+1}(x_1, \ldots, x_m, X_i) = 0}$
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- Remains to define V such that relation search is feasible

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  - Gaudry [G09] : algorithm faster than generic ones for any $q, n \geq 3$ (but still exponential)
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- Idea in both cases : **Weil descent** on Semaev polynomial
  Reduction to a **polynomial system of equations**

Finding relations : Weil descent

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- System harder to solve for larger n
  Attack does not work for $\mathbb{F}_{2^n}$ when n prime

Diem's variant of index calculus [D11b]

Let $K := \mathbb{F}_{2^n}$. Fix $n' < n$ and $m \approx n/n'$

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- **Linear algebra** between the relations

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- See $\mathbb{F}_{2^n}$ as a vector space over $\mathbb{F}_2$
- The polynomial equation over $\mathbb{F}_{2^n}$ corresponds to a **system** of polynomial equations over $\mathbb{F}_2$

Complexity of Diem's algorithm

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  where $T_R(m, n', n)$ is **time to compute one relation**
- (Sparse) linear algebra on relations : cost $2^{\omega' t_3}$ where
$$t_3 \approx \log m + \log n + \omega' n'$$

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- When p is small, systems arising from a Weil descent are **much easier** to solve than random systems
- Under a common **heuristic assumption** validated by experiments for small parameters, we can choose m and n' such that Diem's algorithm for ECDLP over $\mathbb{F}_{2^n}$ has **subexponential** complexity

$$\leq 2^{2n^{2/3}\log n}$$

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- Generic polynomial systems are hard to solve, but **"cryptanalysis" systems are far from generic**
- Systems usually solved with **Gröbner basis algorithms**
- Success stories :
    - HFE and variants
    - Isomorphism of polynomials
    - MacEliece variants
    - Algebraic side-channel attacks

Polynomial systems

- Let K be a field and $R := K[x_1, \ldots, x_n]$.
  Let $f_1, \ldots, f_m \in R$.
  Solve
$$\begin{cases} f_1(x_1, \ldots, x_n) = 0 \\ \ldots \\ f_m(x_1, \ldots, x_n) = 0 \end{cases}$$

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- Linear systems can be solved by triangulation with Gaussian elimination.
  What about polynomial systems ?

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$$g_{i,j} = \sum_k c_{i,j}^k m_k$$

- Write all coefficients in a **Macaulay matrix** $\mathcal{M}_d$, each row corresponding to one polynomial $g_{i,j}$ and each column corresponding to one monomial term m_k

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- The new system is in fact a *Gröbner basis* for the *lexicographic ordering*

Gröbner bases

- Given an ideal $I(f_1, \ldots, f_m)$ and a monomial ordering $>$, a *Gröbner basis* (GB) for this ordering is a basis $\{f_1', \ldots, f_{\ell'}'\}$ such that for any $f \in I(f_1, \ldots f_\ell)$, there exists $i \in \{1, \ldots, \ell'\}$ such that $\mathrm{LT}(f_i')|\mathrm{LT}(f)$ (LT $=$ leading term for the ordering)
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- Any $f \in I$ can be (uniquely) reduced by the GB
- Ideal membership ($f \in I$?) trivial given GB

Gröbner basis algorithms

- First algorithm by Buchberger [B65]
- Connection with linear algebra by Lazard [L83]

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- In F4 and F5, **Macaulay matrices** of increasing size are successively computed and linearly dependent rows are removed with linear algebra until a Gröbner basis is found
- In F5, some rows of the Macaulay matrices are omitted to avoid trivial relations like $0 = f_1 f_2 - f_2 f_1$
- In F4, the reductions are parallelized

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  maximal degree D_{reg} of all polynomials computed
- $\#$ monomials at this degree $\approx n^{D_{reg}}$

- Total cost (n variables) bounded in time and memory by

$$n^{\omega D_{reg}} \qquad \text{and} \qquad n^{2D_{reg}}$$

$\omega \leq 3$ linear algebra constant

"Random" systems

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- *Overdetermined* systems have *lower* degrees of regularity
  Adding new equations helps

Polynomial systems over finite fields

- If $K := \mathbb{F}_q$,
  add the *field equations* $x_i^q - x_i = 0$ to the system

$$\begin{cases} f_1(x_1,\ldots,x_n) &= & 0 \\ & \ldots & \\ f_m(x_1,\ldots,x_n) &= & 0 \\ x_1^q - x_1 &= & 0 \\ & \ldots & \\ x_n^q - x_n &= & 0 \end{cases}$$

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- Degrees of regularity known for “generic” binary systems [BFS04,BFS05]

First fall degree

- Other important parameter : **first fall degree** D_{ff}
  Lowest degree d such that there exist
  non trivial $g_i \in R$ with

$$\max \deg(g_i f_i) = d, \qquad \deg\left(\sum g_i f_i\right) < d$$

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- Sometimes called *degree of regularity* in the literature [DG10,DH11]

Degree of regularity vs. first fall degree

- For many classes of systems :

$$\text{first fall degree } D_{ff} \quad \approx \quad \text{degree of regularity } D_{reg}$$

- Not true in general but **experimental evidence** for “random” systems and many “crypto” systems, including HFE and some variants

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- Intuition : for these systems, there are in fact **many** degree fall relations at D_{ff} or $D_{ff}+1$, that in turn produce many further lower degree relations, etc
- Assumption $D_{ff} \approx D_{reg}$ used in our analysis

Outline

Polynomial systems arising from a Weil descent

- Parameters : n, n', m, t
  $f \in \mathbb{F}_{2^n}[x_1, \ldots x_m]$ with degrees $\leq 2^t - 1$ in all variables
  V a vector subspace of $\mathbb{F}_{2^n}/\mathbb{F}_2$ with dimension n'
- Problem : find $x_i \in V, i = 1, \ldots, m$ such that

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- If $V := \mathbb{F}_{2^n}$, we can use Berlekamp [B70]
- If $mn' \approx n$, we expect ≈ 1 solution

Polynomial systems arising from a Weil descent

- **Weil descent** : if $\{v_1, \ldots, v_{n'}\}$ is a basis of V and $\{\theta_1, \ldots, \theta_n\}$ is a basis of $\mathbb{F}_{2^n}$ over $\mathbb{F}_2$, define **binary variables** $\mathbf{x_{ij}}$ such that $\mathbf{x_i} = \sum_\mathbf{j} \mathbf{x_{ij}} \mathbf{v_j}$

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  define **binary variables** $\mathbf{x_{ij}}$ such that $\mathbf{x_i} = \sum_{\mathbf{j}} \mathbf{x_{ij} v_j}$
  substitute in f and "**reduce modulo** $\mathbf{x_{ij}^2 - x_{ij} = 0}$"
  decompose in the basis $\{\theta_1, \ldots, \theta_n\}$

$$\begin{aligned} 0 &= f(x_1, \ldots, x_m) = f\left(\sum_{j=1}^{n'} x_{1j} v_j, \ldots, \sum_{j=1}^{n'} x_{mj} v_j\right) \\ &= [f]_1^{\downarrow} \theta_1 + \ldots + [f]_n^{\downarrow} \theta_n \end{aligned}$$

- We get n equations $[f]_k^{\downarrow} = 0$ in mn' variables x_{ij}

Degrees and block structure

- If $e = e_0 + e_1 2 + e_2 4 + \ldots + e_{t-1} 2^{t-1}$ then

$$\begin{aligned} x_i^e &= \left(\sum x_{ij} v_j\right)^{e_0} \left(\sum x_{ij}^2 v_j^2\right)^{e_1} \ldots \left(\sum x_{ij}^{2^{t-1}} v_j^{2^{t-1}}\right)^{e_{t-1}} \\ &= \left(\sum x_{ij} v_j\right)^{e_0} \left(\sum x_{ij} v_j^2\right)^{e_1} \ldots \left(\sum x_{ij} v_j^{2^{t-1}}\right)^{e_{t-1}} \end{aligned}$$

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  degree $=$ Hamming weight of $(e_0, \ldots, e_{t-1})$

- $f(x_1, \ldots, x_m) = [f]_1^{\downarrow} \theta_1 + \ldots + [f]_n^{\downarrow} \theta_n$
  Since f has degree at most $2^t - 1$ in each variable x_i,
  Each $[f]_k^{\downarrow}$ has degree at most t
  in each *block of variables* $X_i := \{x_{i1}, \ldots, x_{i,n'}\}$

Applications

- Index calculus for binary elliptic curves
  Semaev's polynomials : degree 2^{m-1} in each variable
- Hidden Field Equation (HFE) polynomial
  degree bounded by D ; quadratic system over $\mathbb{F}_2$
- Index calculus for $\mathbb{F}_{2^n}^*$
  degree 1 in each variable $(t = 1)$
- Factorization problem in $SL(2, \mathbb{F}_{2^n})$
  degree 1 in each variable $(t = 1)$

Example : HFE

- Public Key Cryptosystem proposed by Patarin [P96]
- Private key is a polynomial $f \in \mathbb{F}_{2^n}[x]$
  Public key is a disguised version of its Weil descent
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- Particularities
  - "Disguised" ... but no impact on GB complexity
  - Monovariate ($m = 1$)
  - f has a particular shape

$$f(x) := \sum_{2^i+2^j<D} a_{ij} x^{2^i+2^j} + \sum_{2^i<D} b_i x^{2^i} + c$$

Weil descent on f leads to a *quadratic* system

Back to the general case

- We have n equations in mn' variables x_{ij}, given by

$$0 = f(x_1, \ldots, x_m) = [f]_1^{\downarrow}\theta_1 + \ldots + [f]_n^{\downarrow}\theta_n$$

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- Adding new (low degree) equations may accelerate the resolution
- Can we find more equations ?

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same equations ! (linear combinations)

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  In particular, homogeneous in block X_1
  $f(x_1, \ldots, x_m) = f_0(x_2, \ldots, x_m) + x_1 f_1(x_2, \ldots, x_m)$
  $\Rightarrow x_1 f(x_1, \ldots, x_m) = x_1 f_0(x_2, \ldots, x_m) + x_1^2 f_1(x_2, \ldots, x_m)$

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  $\Rightarrow x_1 f(x_1, \ldots, x_m) = x_1 f_0(x_2, \ldots, x_m) + x_1^2 f_1(x_2, \ldots, x_m)$
- Similar equations with other monomials instead of x_1
  Many new low degree equations

New equations, revisited

- Let $a_{ijk} \in \mathbb{F}_2$ such that $\theta_i\theta_j = \sum_k a_{ijk}\theta_k$

$$x_1 f = \left(\sum_{i=1}^{n} [x_1]_i^{\downarrow} \theta_i\right) \left(\sum_{j=1}^{n} [f]_j^{\downarrow} \theta_j\right) = \sum_{i,j,k=1}^{n} a_{ijk} [x_1]_i^{\downarrow} [f]_j^{\downarrow} \theta_k.$$

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- Hence

$$[x_1 f]_k^{\downarrow} = \sum_{i,j=1}^{n} a_{ijk} [x_1]_i^{\downarrow} [f]_j^{\downarrow} = \sum_{j=1}^{n} p_{ik}(x_{11}, \ldots, x_{1,n'}) [f]_j^{\downarrow}$$

with $\deg(p_{ik}) = 1$

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- The "new" equations $[x_1 f]_k^{\downarrow} = 0$ are *algebraic* combinations of the original ones $[f]_j^{\downarrow} = 0$
- Will be recovered "blindly" by GB algorithms

First fall degree

- We have

$$[x_1 f]_k^{\downarrow} = \sum_{j=1}^{n} p_{ik}(x_{11}, \ldots, x_{1,n'}) \, [f]_j^{\downarrow}$$

$$\deg([x_1 f]_k^{\downarrow}) = mt, \qquad \deg(p_{ik}) = 1, \quad \deg([f]_j^{\downarrow}) = mt$$

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- First fall degree $D_{ff} \leq mt + 1$

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- Non trivial low degree relation !
- First fall degree $D_{ff} \leq mt + 1$
- Essentially as small as it could be (unless f degenerate)

Heuristic assumption

- **We will heuristically assume $\mathbf{D_{reg}} \approx \mathbf{D_{ff}}$**
  in most cases,
  for f chosen randomly with degrees $\leq 2^{t-1}$
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- "Classical" assumption in algebraic cryptanalysis
  - Experimental evidence for "random" and many "crypto" systems **including HFE**
  - (Confusion in literature between the two notions)
- Leads to $\mathbf{D_{reg} \approx mt + 1}$
  (instead of $D_{reg} = n(mt-1)+1$ for generic systems)

Experimental evidence that $D_{reg} \approx mt + 1$

t	n	n'	m	$mt+1$	D_{av}	Av. time (s)	Mem (MB)
1	6	3	2	3	3.1	0	10
1	6	2	3	4	3.8	0	10
1	8	4	2	3	3.0	0	11
1	12	6	2	3	3.6	0	11
1	12	4	3	4	4.2	0	11
1	12	3	4	5	5.3	0	14
1	12	2	6	7	7.4	1	23
1	15	5	3	4	4.1	5	20
1	15	3	5	6	6.3	7	114
1	16	8	2	3	3.0	14	25
1	16	4	4	5	5.3	16	98
1	16	2	8	9	9.6	69	3388
1	18	9	2	3	3.0	85	74
1	18	6	3	4	4.1	86	89
1	18	3	6	7	7.4	233	5398
1	20	10	2	3	3.0	487	291
1	20	5	4	5	6.2	515	733
1	20	4	5	6	6.2	669	3226

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t	n	n'	m	$mt+1$	D_{av}	Av. time (s)	Mem (MB)
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2	12	4	3	7	7.1	1	38
2	12	3	4	9	9.3	2	95
2	15	5	3	7	7.0	12	263
2	16	8	2	5	5.1	13	36
3	6	3	2	7	6.6	0	10
3	12	6	2	7	7.0	1	31
3	12	4	3	10	10.1	9	70
3	12	3	4	13	12.6	70	113
3	15	5	3	10	10.0	118	2371
3	16	8	2	7	7.0	23	253
3	16	4	4	13	13.2	1891	20135
4	8	4	2	9	8.7	1	11
4	12	4	3	13	12.6	199	116
4	15	5	3	13	13.1	2904	6696

Complexity analysis

- Assuming $D_{reg} \approx D_{ff}$, we have $D_{reg} \approx mt + 1$
- Time and memory bounded by

$$n^{\omega D_{reg}} \quad \text{and} \quad n^{2D_{reg}}$$

$\omega \leq 3$: linear algebra constant

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- Block structure $\Rightarrow$ time and memory bounded by

$$(n')^{\omega D_{reg}} \quad \text{and} \quad (n')^{2D_{reg}}$$

Remarks

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- Assumption must be adapted (and checked) in particular cases
- Similar analysis for other "small characteristic" fields

$$D_{reg} \approx (p-1)mt + 1$$

HFE as a particular case

- Cryptanalysis leads to a particular case of our systems with $m = 1, t = \lceil \log_2 D \rceil,\ V = \mathbb{F}_{2^n}$

$$D_{reg} \approx D_{ff} \geq mt + 1 = \lceil \log_2 D \rceil + 1$$

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  We recover [KS99,FJ03,GJS06,DG10,DH11,...]

- No impact of HFE special shape
  Other restrictions may have a (positive) impact [DH11]

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Similarities with HFE

- Polynomial system arising from a Weil descent
- Many low degree relations [C01,...]
- First fall degree [DG10,DH11,...]
- Subsystem with smaller number of variables [GJS06,...] (not discussed here)
- Assumption $D_{reg} \approx D_{ff}$
  widely verified for HFE polynomials [FJ03,GJS06,...]

Outline

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Back to ECDLP

Diem's variant of index calculus [D11b]

Fix $n' < n$ and $m \approx n/n'$

- **Factor basis** :
  Choose a **vector subspace** V of $\mathbb{F}_{2^n}$ with dimension n'
  Define $\mathcal{F}_V := \{(x, y) \in E | x \in V\}$
- **Relation search** : find about $2^{n'}$ relations. For each one,
  Compute $(X_i, Y_i) := a_i P + b_i Q$ for random a_i, b_i
  Find $x_j \in V$ with $S_{m+1}(x_1, \ldots, x_m, X_i) = 0$
  Find the corresponding y_j
- **Linear algebra** between the relations

Finding relations

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- Weil descent $\rightarrow$ polynomial system
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  - m variables
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- ! Summation polynomials not "random" ! (symmetric,...)

Heuristic assumption

- Let n, n', m, E be fixed.
  Let $R_i = (X_i, Y_i)$ be a random point of E.
  Let V be a random vector space of dimension n'.

- **Assumption** : after applying a Weil descent to

$$S_{m+1}(x_1, \ldots, x_m, X_i) = 0,$$

  the resulting system satisfies $\mathbf{D_{reg}} \approx \mathbf{D_{ff}}$

Experimental verification $\mathbf{D_{reg}} \approx \mathbf{D_{ff}}$

- Random curves $E : y^2 + xy = x^3 + a_4 x^2 + a_6$ for random a_4, a_6

n	n'	m	t	$mt+1$ ($\geq D_{ff}$)	D_{av}	Time	Mem.
11	6	2	2	5	3.0	0	11
11	4	3	3	10	7.1	1	15
17	9	2	2	5	4.0	0	16
17	6	3	3	10	7.1	130	2136

D_{reg} even *lower* than expected

Experimental verification $\mathbf{D_{reg}} \approx \mathbf{D_{ff}}$

- Koblitz curves $E : y^2 + xy = x^3 + x^2 + 1$

n	n'	m	t	$mt + 1$ ($\geq D_{ff}$)	D_{av}	Time	Mem.
11	6	2	2	5	3.0	0	11
11	4	3	3	10	7.1	1	15
17	9	2	2	5	4.0	0	15
17	6	3	3	10	7.2	132	2133

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Complexity of Diem's algorithm

- Computing S_{m+1} with resultants : cost 2^{t_1} where

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$$t_2 \approx n' + m \log m + \omega(m^2+1) \log n'$$

  - Each one costs $(n')^{\omega(mt+1)} = (n')^{\omega(m^2+1)}$
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- (Sparse) linear algebra on relations : cost $2^{\omega' t_3}$ where

$$t_3 \approx \log m + \log n + \omega' n'$$

Estimations for "small" parameters

n	m	n'	t_1	t_2	t_3	t_{max}
50	2	25	6	97	57	97
100	2	50	6	137	108	137
160	2	80	6	177	168	177
200	2	100	6	202	209	209
500	3	167	12	393	344	393
1000	4	250	20	664	512	664
2000	4	500	20	965	1013	1013
5000	6	833	42	1926	1682	1926
10000	7	1429	56	3020	2873	3020
20000	9	2222	90	4986	4462	4986
50000	11	4545	132	9030	9110	9110
100000	14	7143	210	14762	14306	14762

Asymptotic estimates

- Fix $n' := n^\alpha$ and $m := n^{1-\alpha}$ for $\alpha := 2/3$

$$
\begin{aligned}
t_1 &\approx n^{2/3}, \\
t_2 &\approx (1/3)n^{1/3}\log n + n^{2/3} + (2/3)\omega n^{2/3}\log n, \\
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\end{aligned}
$$

- Overall complexity

$$
2^T \quad \text{with} \qquad T \approx cn^{2/3}\log n \quad \text{and} \quad c := \frac{2}{3}\omega \leq 2
$$

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Conclusion

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  - Reasonable evidence under heuristic assumption
  - Diem's algorithm would beat BSGS for $n \geq 2000$
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  - Very important class of systems for cryptography
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- Polynomial systems arising from a Weil descent
  - Very important class of systems for cryptography
  - ECDLP, HFE, DLP, factoring in $SL(2, \mathbb{F}_{2^n})$, ...
- Future work
  - Better algorithms, remove heuristic assumptions
  - Extension to prime fields ?

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